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二项式系数与 Lucas 数 4q 次幂的关系

陈小芳

(渭南师范学院 数学与统计学院,陕西 渭南 714099)

摘　要:结合 Lucas 数列与二项式系数的定义及性质,利用初等数论的方法研究了二项式系数与 Lucas 数的 4q 次幂的卷积并给出了证明,得到了二项式系数与 Lucas 数 4q 次幂之间的恒等式关系,推广了已有的组合数与 Lucas 数的较小正整数次幂的恒等式相关结果.

关键词:Lucas 数;4q 次幂;二项式系数

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Relation between binomial coefficients and 4q-power of Lucas numbers

CHEN Xiao-fang

(School of Mathematics and Statistics, Weinan Normal University, Weinan 714099, China)

Abstract: According to the definition and the properties of the Lucas sequence and combinatorial numbers, the identity relationship between the binomial coefficient and 4q-powers of the Lucas numbers is proved by using methods of elementary number theory, thus generalizing the related results of identity of the lower positive integer power of combinatorial number and Lucas number.

Keywords: Lucas number; 4q-power; convolution; binomial coefficient

对于整数 n 和 i,$n \geqslant i \geqslant 0$,$\binom{n}{i}=\frac{n!}{i!\ (n-i)!}$,其中$\binom{n}{i}$是正整数,表示从 n 个互不相同的元素中任选 i 个元素的组合个数,结合二项式定理可得

$$(1+x)^n=\sum_{i=0}^{n}\binom{n}{i}x^i \qquad (1)$$

,

其中 x 为任意复数,故称$\binom{n}{i}$,$n \geqslant i \geqslant 0$ 为二项式系数,记为 C_n^i.

Lucas 数列是数论中重要的一个数列,由递推公式 $L_{l+1}=L_l+L_{l-1}(n=1,2,\cdots)$, $L_0=2, L_1=1$ 所确定的数列被称为 Lucas 数列. 众多学者致力于二项式系数和 Lucas 数列的各种性质的研究,得到了许多关于二项式系数的恒等式和 Lucas 数列的恒等式[1-15].

对于给定的非负整数 k 以及正整数 m 和 n,定义数列$\left\{C_n^i\right\}_{i=0}^{n}$和$\left\{L_{k+i}^m\right\}_{i=0}^{n}$的卷积 $l(k,m,n)$ 为

$$l(k,m,n)=C_n^0L_k^m+C_n^1L_{k+1}^m+\cdots+C_n^mL_{k+n}^m=\sum_{i=0}^{n}C_n^iL_{k+i}^m. \qquad (2)$$

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作者简介:陈小芳(1978-),女,汉族,陕西渭南人,副教授,研究方向:数论. Email:50338962@ qq. com

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对于 $l(k,m,n)$ 的计算,晁晶晶,陈小芳等分别给出了 $m=1$、$m=2$ 和 $m=3$ 的情形[1-3],证明了

$$l(k,1,n)=2^k L_{n+2k}. \tag{3}$$

$$l(k,2,n)=\begin{cases}5^{\frac{n+1}{2}}\times F_{2k+n} & n=2l+1,\\ 5^{\frac{n}{2}}\times L_{2k+n} & n=2l.\end{cases} \tag{4}$$

$$l(k,3,n)=\begin{cases}2^n L_{3k+2n}+3(-1)^{k+n}L_{k-n} & k\ge n,\\ 2^n L_{3k+2n}+3L_{n-k} & k<n.\end{cases} \tag{5}$$

其中的 L_{n+2k} 和 F_{2k+n} 分别为第 $n+2k$ 个 Lucas 数和第 $2k+n$ 个 Fibonacci 数,L_{3k+2n} 和 L_{n-k} 分别表示第 $3k+2n$ 和第 n-k 个 Lucas 数.

1 结论

本文利用初等数论方法研究了当 $m=4q\ (q\in N^*)$ 时 $l(k,m,n)$ 的情形,得到了关于二项式系数与 Lucas 数 $4q$ 次幂的一个恒等式,证明了下面的定理.

定理 对于 $k\in N$,且 $q,n\in N^*$,定义 $l(k,m,n)$ 二项式系数 $\left\{C_n^i\right\}_{i=0}^n$ 和 Lucas 数 $\left\{L_{k+i}^m\right\}_{i=0}^n$ 的卷积(1),则当 $m=4q\ (q\in N^*)$ 时

$$\begin{aligned}
l(k,4q,n)=&C_n^0L_k^{4q}+C_n^1L_{k+1}^{4q}+\cdots+C_n^nL_{k+n}^{4q}=\\
&L_{2q}^nL_{4qk+2qn}+C_{4q}^1(-1)^k\Big(L_{2q-1}^nL_{(4q-2)k+(2q-1)n}+2C_n^1L_{2q-1}^{n-1}L_{(4q-2)k+(2q-1)(n-1)}+\\
&2^2C_n^2L_{2q-1}^{n-2}L_{(4q-2)k+(2q-1)(n-2)}+\cdots+2^{n-1}nL_{(4q-2)k+(2q-1)}L_{2q-1}+2^n\Big)+\\
&C_{4q}^2L_{2q-2}^nL_{(4q-4)k+(2q-2)n}+C_{4q}^3(-1)^k\Big(L_{2q-3}^nL_{(4q-6)k+(2q-3)n}+2C_n^1L_{2q-3}^{n-1}L_{(4q-2)k+(2q-1)(n-1)}+\\
&2^2C_n^2L_{2q-3}^{n-2}L_{(4q-2)k+(2q-1)(n-2)}+\cdots+2^{n-1}nL_{(4q-2)k+(2q-1)}L_{2q-3}+2^n\Big)+\\
&\cdots+\\
&C_{4q}^{2q-1}(-1)^k\Big(L_1^nL_{2k+n}+2C_n^1L_1^{n-1}L_{2k+n-1}+2^2C_n^2L_1^{n-2}L_{2k+n-2}+\cdots+2^{n-1}nL_1L_{2k+1}+2^n\Big)+2^n.
\end{aligned} \tag{6}$$

其中的 L_{2q},L_{2q-1},L_{2q-2},$\cdots$ 及 $L_{4qk+2qn}$,$L_{(4q-2)+(2q-1)n}$,$L_{(4q-4)+(2q-2)n}$,$\cdots$ 分别为第 $2q$,$2q$-1,$2q$-2,$\cdots$ 及第 $4qk+2qn$,$(4q$-$2)+(2q$-$1)n$,$(4q$-$4)+(2q$-$2)n$,$\cdots$ 个 Lucas 数.

2 定理的证明

已知 $L_n=\alpha^n+\beta^n$. (7)

其中 $\alpha=\dfrac{1+\sqrt{5}}{2},\beta=\dfrac{1-\sqrt{5}}{2}$. (8)

当 $m=4q(q\in N^*)$ 时,由(1)及(7)可得

$$
\begin{aligned}
L_k^{4q} &= \left(\alpha^k+\beta^k\right)^{4q} = C_{4q}^0(\alpha^k)^0\cdot(\beta^k)^{4q} + C_{4q}^1(\alpha^k)^1\cdot(\beta^k)^{4q-1} + C_{4q}^2(\alpha^k)^2\cdot(\beta^k)^{4q-2} + \\
&\cdots + C_{4q}^{4q-2}(\alpha^k)^{4q-2}\cdot(\beta^k)^2 + C_{4q}^{4q-1}(\alpha^k)^{4q-1}\cdot(\beta^k)^1 + C_{4q}^{4q}(\alpha^k)^{4q}\cdot(\beta^k)^0 = \\
&\alpha^{4qk}+\beta^{4qk}+C_{4q}^1\alpha^k\cdot\beta^k\cdot\beta^{(4q-2)k}+C_{4q}^1\alpha^k\cdot\beta^k\cdot\alpha^{(4q-2)l}+C_{4q}^2\alpha^{2k}\cdot\beta^{2k}\cdot\beta^{(4q-4)k}+ \\
&C_{4q}^2\alpha^{2k}\cdot\beta^{2k}\cdot\alpha^{(4q-2)k}+C_{4q}^3\alpha^{3k}\cdot\beta^{3k}\cdot\beta^{(4q-6)k}+C_{4q}^3\alpha^{3k}\cdot\beta^{3k}\cdot\alpha^{(4q-6)k}+\cdots+ \\
&C_{4q}^{4q-1}\alpha^{(2q-1)k}\cdot\beta^{(2q-1)k}\cdot\beta^{[4q-2(2q-1)]k}+C_{4q}^{4q-1}\alpha^{(2q-1)k}\cdot\beta^{(2q-1)k}\cdot\alpha^{[4q-2(2q-1)]k}+C_{4q}^{2q}.
\end{aligned}
$$

由(8)知 $\alpha\beta=-1$,

$$
\begin{aligned}
L_k^{4q} &= \alpha^{4qk}+\beta^{4qk}+C_{4q}^1(-1)^k\cdot\beta^{(4q-2)k}+C_{4q}^1(-1)^k\cdot\alpha^{(4q-2)k}+C_{4q}^2\beta^{(4q-4)k}+C_{4q}^2\alpha^{(4q-2)k}+ \\
&C_{4q}^2(-1)^k\cdot\beta^{(4q-6)k}+C_{4q}^2(-1)^k\cdot\alpha^{(4q-6)k}+\cdots+C_{4q}^{2q-1}(-1)^k\cdot\beta^{2k}+C_{4q}^{2q-1}(-1)^k\cdot\alpha^{2k}+C_{4q}^{2q}.
\end{aligned} \tag{9}
$$

将(2)与(9)结合可知

$$
\begin{aligned}
l(k,4q,n) &= \sum_{i=0}^{n} C_n^i L_{k+i}^{4q} \\
&= C_n^0\Big(\alpha^{4qk}+\beta^{4qk}+C_{4q}^1(-1)^k\cdot\beta^{(4q-2)k}+C_{4q}^1(-1)^k\cdot\alpha^{(4q-2)k}+C_{4q}^2\beta^{(4q-4)k}+C_{4q}^2\alpha^{(4q-2)k}+ \\
&\quad\cdots+C_{4q}^{2q-1}(-1)^k\cdot\beta^{2k}+C_{4q}^{2q-1}(-1)^k\cdot\alpha^{2k}+C_{4q}^{2q}\Big)+ \\
&C_n^1\Big(\alpha^{4q(k+1)}+\beta^{4q(k+1)}+C_{4q}^1(-1)^k\cdot\beta^{(4q-2)(k+1)}+C_{4q}^1(-1)^k\cdot\alpha^{(4q-2)(k+1)}+C_{4q}^2\beta^{(4q-4)(k+1)}+ \\
&C_{4q}^2\alpha^{(4q-4)(k+1)}+\cdots+C_{4q}^{2q-1}(-1)^k\cdot\beta^{2(k+1)}+C_{4q}^{2q-1}(-1)^k\cdot\alpha^{2(k+1)}+C_{4q}^{2q}\Big)+ \\
&\quad\cdots+ \\
&C_n^{n-1}\Big(\alpha^{4q(k+n-1)}+\beta^{4q(k+n-1)}+C_{4q}^1(-1)^k\cdot\beta^{(4q-2)(k+n-1)}+C_{4q}^1(-1)^k\cdot\alpha^{(4q-2)(k+n-1)}+ \\
&C_{4q}^2\alpha^{(4q-4)(k+n-1)}+C_{4q}^2\alpha^{(4q-2)(k+n-1)}+\cdots+C_{4q}^{2q-1}(-1)^k\cdot\beta^{2(k+n-1)}+C_{4q}^{2q-1}(-1)^k\cdot\alpha^{2(k+n-1)}+C_{4q}^{2q}\Big)+ \\
&C_n^n\Big(\alpha^{4q(k+n)}+\beta^{4q(k+n)}+C_{4q}^1(-1)^k\cdot\beta^{(4q-2)(k+n)}+C_{4q}^1(-1)^k\cdot\alpha^{(4q-2)(k+n)}+C_{4q}^2\beta^{(4q-4)(k+n)}+ \\
&C_{4q}^2\alpha^{(4q-2)(k+n)}+\cdots+C_{4q}^{2q-1}(-1)^k\cdot\beta^{2(k+n)}+C_{4q}^{2q-1}(-1)^k\cdot\alpha^{2(k+n)}+C_{4q}^{2q}\Big)= \\
&\alpha^{4kq}\sum_{i=0}^{n}C_n^i\alpha^{4qi}+\beta^{4kq}\sum_{i=0}^{n}C_n^i\beta^{4qi}+C_{4q}^1(-1)^k\alpha^{(4q-2)k}\sum_{i=0}^{n}C_n^i\alpha^{(4q-2)i}+ \\
&C_{4q}^1(-1)^k\beta^{(4q-2)k}\sum_{i=0}^{n}C_n^i\beta^{(4q-2)i}+C_{4q}^2\alpha^{(4q-4)k}\sum_{i=0}^{n}C_n^i\alpha^{(4q-4)i}+C_{4q}^2\beta^{(4q-4)k}\sum_{i=0}^{n}C_n^i\beta^{(4q-4)i}+ \\
&\cdots+C_{4q}^{2q-1}(-1)^k\beta^{2k}\sum_{i=0}^{n}C_n^i\beta^{2i}+C_{4q}^{2q-1}(-1)^k\alpha^{2k}\sum_{i=0}^{n}C_n^i\alpha^{2i}+C_{4q}^{2q}\cdot 2^n.
\end{aligned} \tag{10}
$$

又由二项式定理可知

$$\begin{aligned}
&\sum_{i=0}^{n} C_n^i \alpha^{4qi} = (1+\alpha^{4q})^n, && \sum_{i=0}^{n} C_n^i \beta^{4qi} = (1+\beta^{4q})^n,\\
&\sum_{i=0}^{n} C_n^i \alpha^{(4q-2)i} = (1+\alpha^{4q-2})^n, && \sum_{i=0}^{n} C_n^i \beta^{(4q-2)i} = (1+\beta^{4q-2})^n,\\
&\sum_{i=0}^{n} C_n^i \alpha^{(4q-4)i} = (1+\alpha^{4q-4})^n, && \sum_{i=0}^{n} C_n^i \beta^{(4q-4)i} = (1+\beta^{4q-4})^n,\\
&\qquad\qquad\vdots\\
&\sum_{i=0}^{n} C_n^i \alpha^{4i} = (1+\alpha^{4})^n, && \sum_{i=0}^{n} C_n^i \beta^{4i} = (1+\beta^{4})^n,\\
&\sum_{i=0}^{n} C_n^i \alpha^{2i} = (1+\alpha^{2})^n, && \sum_{i=0}^{n} C_n^i \beta^{2i} = (1+\beta^{2})^n,
\end{aligned} \tag{11}$$

结合(8)与(11)得

$$\begin{aligned}
&1+\alpha^{4q}=L_{2q}\alpha^{2q}, 1+\beta^{4q}=L_{2q}\beta^{2q},\\
&1+\alpha^{4q-2}=L_{2q-1}\alpha^{2q-1}+2,\ 1+\beta^{4q-2}=L_{2q-1}\beta^{2q-1}+2,\\
&1+\alpha^{4q-4}=L_{2q-2}\alpha^{2q-2},\ 1+\beta^{4q-4}=L_{2q-2}\beta^{2q-2},\\
&\qquad\cdots\\
&1+\alpha^{4}=L_{2}\alpha^{2},\ 1+\beta^{4}=L_{2}\beta^{2},\\
&1+\alpha^{2}=L_{1}\alpha+2,\ 1+\beta^{2}=L_{1}\beta+2.
\end{aligned} \tag{12}$$

将(11)和(12)代入(10)有

$$\begin{aligned}
l(k,4q,n) = {}& \alpha^{4kq}\left(1+\alpha^{4q}\right)^n + \beta^{4kq}\left(1+\beta^{4q}\right)^n + C_{4q}^{1}(-1)^k \alpha^{(4q-2)k}\left(1+\alpha^{4q-2}\right)^n +\\
& C_{4q}^{1}(-1)^k \beta^{(4q-2)k}\left(1+\beta^{4q-2}\right)^n + C_{4q}^{2}\alpha^{(4q-4)k}\left(1+\alpha^{4q-4}\right)^n +\\
& C_{4q}^{2}\beta^{(4q-4)k}\left(1+\beta^{4q-2}\right)^n + \cdots + C_{4q}^{2q-1}(-1)^k \beta^{2k}\left(1+\beta^{2}\right)^n +\\
& C_{4q}^{2q-1}(-1)^k \alpha^{2k}\left(1+\alpha^{2}\right)^n + C_{4q}^{2q}\cdot 2^n =\\
& \alpha^{4kq}\left(L_{2q}\alpha^{2q}\right)^n + \beta^{4kq}\left(L_{2q}\beta^{2q}\right)^n + C_{4q}^{1}(-1)^k \alpha^{(4q-2)k}\left(L_{2q-1}\alpha^{2q-1}+2\right)^n +\\
& C_{4q}^{1}(-1)^k \beta^{(4q-2)k}\left(L_{2q-1}\beta^{2q-1}+2\right)^n + C_{4q}^{2}\alpha^{(4q-4)k}\left(L_{2q-2}\alpha^{2q-2}\right)^n +\\
& C_{4q}^{2}\beta^{(4q-4)k}\left(L_{2q-2}\beta^{2q-2}\right)^n + \cdots + C_{4q}^{2q-1}(-1)^k \beta^{2k}\left(L_1\beta+2\right)^n\\
& C_{4q}^{2q-1}(-1)^k \alpha^{2k}\left(L_1\alpha+2\right)^n + C_{4q}^{2q}\cdot 2^n.
\end{aligned}$$

整理化简可得恒等式

$$l(k,4q,n)=L_{2q}^{n}L_{4qk+2qn}+C_{4q}^{1}(-1)^{k}\Big(L_{2q-1}^{n}L_{(4q-2)k+(2q-1)n}+2C_{n}^{1}L_{2q-1}^{n-1}L_{(4q-2)k+(2q-1)(n-1)}+$$
$$2^{2}C_{n}^{2}L_{2q-1}^{n-2}L_{(4q-2)k+(2q-1)(n-2)}+\cdots+2^{n-1}nL_{(4q-2)k+(2q-1)}L_{2q-1}+2^{n}\Big)+$$
$$C_{4q}^{2}L_{2q-2}^{n}L_{(4q-4)k+(2q-2)n}+C_{4q}^{3}(-1)^{k}\Big(L_{2q-3}^{n}L_{(4q-6)k+(2q-3)n}+2C_{n}^{1}L_{2q-3}^{n-1}L_{(4q-2)k+(2q-1)(n-1)}+$$
$$2^{2}C_{n}^{2}L_{2q-3}^{n-2}L_{(4q-2)k+(2q-1)(n-2)}+\cdots+2^{n-1}nL_{(4q-2)k+(2q-1)}L_{2q-3}+2^{n}\Big)+$$
$$\cdots+$$
$$C_{4q}^{2q-1}(-1)^{k}\Big(L_{1}^{n}L_{2k+n}+2C_{n}^{1}L_{1}^{n-1}L_{2k+n-1}+2^{2}C_{n}^{2}L_{1}^{n-2}L_{2k+n-2}+\cdots+2^{n-1}nL_{1}L_{2k+1}+2^{n}\Big)+2^{n}.$$

定理得证.

3 结束语

对于任意给定的正整数 q,如何证明得出的二项式系数与 Lucas 数 q 次幂的恒等关系式,有待于进一步研究.

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